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Unpacking Middle School Students' Ideas about Perimeter: A Case Study of Mathematical Discourse in the Classroom

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Although decades of mathematics education reform supports using effective classroom discourse to increase students' learning of mathematics, research about what mathematics students learn in such classrooms is less developed. Moreover, how teachers actually facilitate classroom discourse and navigate through unpredictable terrain to develop students' understanding of mathematics remains challenging for teachers. In this paper, I examine the practice of an experienced middle school teacher as she leads her students in classroom discussions about perimeter of rectangles. Unpacking how she facilitates discourse in the classroom and addresses her students' mathematical ideas about perimeter shows how students' insights or misconceptions are identified and clarified, explained and illustrated, tested, and then affirmed or revised. Key findings of this case study include how the teacher uses a specific curriculum unit and divergent questions to support dialogic mathematical discourse, how she addresses a misconception, how she fosters a mathematical disciplinary community in the classroom, and how she copes with time constraints. Implications of these findings are discussed.

Introduction

"*The teacher has a central role in orchestrating the oral and written discourse in ways that contribute to students' understanding of mathematics*" (National Council of Teachers of Mathematics [NCTM], 1991, p. 35).

"*Teachers must also decide … how to organize and orchestrate the work of students, what questions to ask to challenge those with varied levels of expertise, and how to support students without taking over the process of thinking for them and thus eliminating the challenge*" (NCTM, 2000, p. 19).

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"*Teachers can effectively use questions during the whole class discussion to help students clarify their thinking and to challenge them to think more deeply about the ideas presented … Students' understanding and conceptions are either refined or changed as they reflect on questions that are posed by the teacher*" (Lamberg, 2013, p. 96).

For decades, leaders of mathematics education reform have emphasized that teachers should facilitate and support students in sharing their ideas about mathematics to assess prior knowledge and learning, develop understanding of new concepts, and grow as a community of learners (cf., Ball, 1990; Herbel-Eisenmann & Otten, 2011; NCTM 1991, 2000). However, how discourse is defined by researchers varies and how teachers implement discourse in the classroom is variable in effectiveness. Yet substantial evidence demonstrates that discourse in the mathematics classroom can be powerful in supporting students' learning, especially when coupled with curricula which seeks to implement the NCTM *Standards* (e.g., Clarke, 1997; Conklin, Grant, Ludema, Rickard, & Rivette, 2006; Ryve, 2011; Truxaw & DeFranco, 2008). Responding to the recommendations of reform movements and the potential of mathematical discourse for students, mathematics educators have produced curricula and professional development materials aimed at supporting teachers in implementing mathematical discourse in their classrooms so that sharing and talking about mathematics becomes routine (e.g., Lamberg, 2013; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002; Shroyer & Fitzgerald, 1986; Stein, Smith, Henningsen, & Silver, 2000). Supporting teachers and students in sharing and discussing mathematics is also central to some culturally based mathematics curricula. Such curricula are specifically designed to increase mathematics achievement of historically underserved indigenous students (e.g., American Indian and Alaska Native students) and have been shown to be effective in increasing students' mathematics achievement (e.g., Kagle, Barber, Lipka, Sharp, & Rickard, 2007; Kisker, Lipka, Adams, Rickard, Andrew-Irke, Yanez, & Millard, 2012; Legaspi & Rickard, 2005; Lipka, Jones, Gilsdorf, Remick, & Rickard, 2010).

Although writers of mathematics education reforms and innovative K-12 mathematics curricula have been recommending and providing support for mathematical discourse in the classroom for over 20 years (e.g., Lappan et al., 2002; Shroyer & Fitzgerald, 1986), research on what teachers themselves need to know in order to orchestrate classroom discourse effectively in the mathematics classroom is more recent. One line of research has provided strong evidence that teachers need to not only have deep and connected subject matter knowledge of mathematics to lead mathematical discussions with their students, but also need to know mathematics content that is specific to teaching and specifically addresses the needs of learners (Hill, Rowan, & Ball, 2005).

Another key area identified and explored by researchers has been that teachers need to possess dispositions which empower them to be open to talking about and exploring mathematics in flexible and often unpredictable ways with students to implement NCTM Standards-based curricula, which includes effectively leading mathematical classroom discourse (Ross, McDougall, Hogaboam-Gray, & LeSage, 2003). In other words, to support students' learning in the classroom through mathematical discourse, teachers need to know the mathematics being taught in ways that can help students make sense of it. They also need to be open to and skilled in talking with students about the mathematics on the students' terms so they may reason with and unpack key mathematical ideas (Rickard, 2005a, 2005b). Such knowledge of mathematics for teaching and dispositions can enable teachers, for example, to "…help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or dispute these conjectures" (NCTM, 2000, p. 3). Students developing, sharing, testing, and refining or revising conjectures has long been recognized as a key aspect of mathematics discourse in the classroom (e.g., Ball, 1990; Rickard, 1995, 1996, 1998) and is an authentic disciplinary activity in doing mathematics (Lakatos, 1995; Polya, 1957; Schoenfeld, 1985).

Recent research on mathematical discourse has demonstrated that "discourse" in mathematics education

research may refer to communication and interaction between teacher and students in the classroom, constructs and/or tools of analysis about discourse borrowed from other disciplines, or conceptualizing mathematics itself as a discourse (Ryve, 2011). As is predominant in K-12 reform documents (e.g., NCTM, 1991, 2000), this paper focuses on mathematical discourse in the classroom as the process of how teachers and students communicate and interact together as they teach and learn mathematics. Furthermore, the nature of mathematical discourse in the classroom that is the focus of this paper is dialogic, which "… is characterized by give-and-take communication that uses dialogue as a process for thinking" (Truxaw & DeFranco, 2008, p. 489). In this case study, dialogic discourse is an interactive conversation and exchange of mathematical ideas between and among the teacher and students that is facilitated by the teacher to support students' learning. Supporting dialogic discourse in the mathematics classroom is aligned with reforms and research findings that dialogic discourse is correlated with conceptual understanding (e.g., see Truxaw & DeFranco, 2008).

The above findings about dialogic mathematical discourse are consistent with research on the powerful role that verbal interaction between teacher and students plays in the learning process (e.g., Vygotsky, 1978) as well as evidence that, by shaping the mathematical discourse of the classroom, teachers can shape student learning (e.g., Ball, 1990). Moreover, from the perspective of mathematics as a discipline, dialogic mathematical discourse is part of how mathematical discovery occurs, with ideas being shared, discussed, tested, and revised in the disciplinary community of mathematics (Lakatos, 1995). Intertwining these two threads of research provides the key theoretical assumption for this case study: Students can learn mathematics with deeper conceptual understanding by interacting with their teacher and peers via dialogic mathematical discourse. Engaging in dialogic mathematical discourse means that the teacher facilitates how students develop, explain, test, and revise their ideas about mathematical concepts, similar to how constructing mathematical ideas occurs in the discipline of mathematics and

as is advocated in mathematics education reforms (cf., Ball, 1990; Bruner, 1960; Lamberg, 2013; NCTM, 2000).

The focus of this case study is to investigate how a middle school mathematics teacher shapes her students' learning by researching her practice through the above theoretical lens. Specifically, how and to what extent does she employ dialogic mathematical discourse in her classroom to help students learn a mathematical concept in ways that are consistent with mathematics education reforms and parallel the activities that involve creating mathematics as in the disciplinary community of mathematics? Additionally, this research aims to identify and unpack pedagogical challenges and opportunities that occur in the teacher's classroom, explore how she addresses these with her students, and ask her to reflect on her teaching to understand her thinking about these key decisions. This research has the potential to inform other mathematics teachers in developing their own practices, particularly use of dialogic mathematical discourse in the classroom, to improve students' learning and implement mathematics education reforms.

Background and Methodology

Participant

Ellen Wilson (a pseudonym) is a sixth-grade teacher with 17 years of teaching experience, all at the middle school level (i.e., grades 6-8). Ellen has been teaching at her middle school for 7 years, with the other 10 years of her experience at another middle school in the same district. As part of her undergraduate studies, Ellen completed a college algebra course and a two-semester course sequence on mathematics for elementary teachers. Ellen is widely regarded by her colleagues and the school administration as being a skilled and effective teacher in all subject areas, including mathematics. As well as mathematics, she has also taught English, reading, and social studies.

During this study, Ellen taught two general mathematics classes and one class of enriched mathematics, all at the sixth-grade level. As Ellen explained about her mathematics classes, "The difference between my students in general and enriched

math is that the general math kids have tested at grade level and the enriched kids have tested above grade level" on the school district's math assessment administered at the end of the fifth grade. Ellen's district has adopted a sixth-grade textbook for the general mathematics classes and uses a seventh-grade text from the same series for the enriched mathematics classes. Ellen described that she uses the texts in both classes as a "kind of resource or reference" but uses supplementary mathematics curricula and activities she has developed herself for "about half" of her instruction. The class that participated in this case study is one of Ellen's general mathematics classes. Of her 21 students, 14 are girls and 7 are boys. Reflecting the diversity of Ellen's school[1], 13 students are Caucasian, 5 are African American, 2 are Hispanic, and 1 is Asian. Ellen has arranged students' desks in her classroom into clusters of four to facilitate small-group work and collaboration.

Ellen and I began our work together as part of a school-university collaboration where teachers at Ellen's school work with university-based faculty and graduate students. Ellen volunteered to be a part of this collaboration, particularly seeing the collaboration as a way of developing her practice further around teaching and learning middle school mathematics in ways consistent with the NCTM *Standards*. After having several preliminary meetings where we reviewed and discussed the NCTM *Standards* (e.g., NCTM, 2000) and examples of reform-oriented curricula (e.g., Shroyer & Fitzgerald, 1986), I sat in on a few of Ellen's lessons in her general math class described above, which included her teaching several activities from a *Standards*-based middle school mathematics unit[2]. It was during these lessons that Ellen decided she wanted to focus on "talking about mathematics" (i.e., dialogic mathematical discourse) with her students. In particular, Ellen wanted to learn more about facilitating discussions in teaching mathematics to better engage her students. Ellen noted that she already routinely used interactive discussions in teaching her reading and English classes. She felt that it worked well and wanted to expand this aspect of her practice to include mathematics. As a result, we agreed that our collaboration would include studying her practice over several weeks of teaching about perimeter and

area, and using the resulting data to improve our understanding of mathematical discourse in her classroom.

Based on our preliminary discussions and review of curriculum materials, Ellen decided that she would teach the middle school *Mouse and Elephant*: *Measuring Growth* unit[3] (Shroyer & Fitzgerald, 1986) on perimeter, area, surface area, and volume. Ellen teaching the *Mouse and Elephant* unit is consistent with the theoretical assumptions underlying this case study – i.e., the teacher facilitating mathematical discourse with her students so they can develop deep understanding of key concepts (e.g., perimeter) as a disciplinary community of learners. Importantly, the *Mouse and Elephant* unit provides extensive support to teachers for facilitating classroom discussions with students about the mathematics. For each activity in *Mouse and Elephant*, the teacher is provided with a script divided into three columns, which are teacher action, teacher talk, and expected response, respectively (Shroyer & Fitzgerald, 1986, p. 2):

- Teacher Action: This column includes materials used, what to display on the overhead, when to explain a concept, what to ask a questions, etc.
- Teacher Talk: This column includes important questions and explanations that are needed to develop understandings and problem-solving skills, etc.
- Expected Response: This column includes correct responses as well as frequent incorrect responses and suggestions for handling them.

The script in *Mouse and Elephant*, therefore, is intended to provide teachers with a detailed guide about how to teach each activity, including what material to use (e.g., display on overhead and/or handout copies of the appropriate black line masters supplied in the unit), how to pose key mathematical ideas and problems to students, solutions, and suggestions for addressing students' misconceptions or errors. In other words, the unit supports dialogic mathematical discourse in the classroom in which the teacher and her students interact with each other to develop and refine new understandings of mathematics together (Truxaw & DeFranco, 1986). *Mouse and*

Elephant is also clearly aligned with the NCTM *Standards* (cf., NCTM 1991, 2000; Shroyer & Fitzgerald, 1986).

Case Study Design

The structure for this research of Ellen's practice utilizes case study methodology. Case study methods are appropriate for researching individuals and their particular context (Bogdan & Biklen, 1982). As an observational case study, data collection included audio taping the preliminary meetings that Ellen and I had together, and portions of these tapes were transcribed for analysis. Additional data was collected during daily lessons I observed Ellen teach *Mouse and Elephant* over a three-week period. All lessons were audio taped and I recorded field notes during each lesson as well. Each set of field notes was then expanded, usually the same day, in a comprehensive journal kept throughout the study. The journal includes personal reflections and preliminary data analysis. Documents were also used as sources of data, including articles and materials reviewed and discussed by Ellen and I during our preliminary meetings, Ellen's lesson plans and curriculum materials which she used, and samples of students' work. After most lessons, Ellen and I also had a brief (usually 10-12 minutes) conversation to debrief and reflect on the lesson, during which Ellen shared her thoughts and reflections on the lesson. I could also ask follow up questions. Each of these conversations was audio taped and fully transcribed for analysis (see Bogdan & Biklin, 1982, for additional information about collecting, organizing, and identifying patterns of practice in educational settings from multiple sources of qualitative data).

Data Analysis

In analyzing the collected data, I focused on patterns in Ellen's practice that I observed in her classroom teaching, trying to identify and understand those features that seemed critical in supporting dialogic mathematical discourse – e.g., features that pushed her students to develop, share, discuss, and revise their ideas as occurs in the discipline of mathematics.

Additionally, patterns of practice observed in Ellen's classroom over the three weeks of this case study were triangulated with students' work (e.g., class discussions, homework assignments) and follow-up conversations with Ellen in which she shared her intent in planning the particular lesson, unpacking how she felt the lesson went, and reflecting on why she made particular decisions. Throughout the three weeks of data collection for this case study, both Ellen and I paid close attention to how she was helping her students talk about mathematics. In particular, how Ellen and her students talked about mathematics during *Mouse and Elephant* was our shared central research interest. This focus is supported by our theoretical assumptions about students learning mathematics through interaction and conversation, intended to create a classroom environment that incorporated aspects of a disciplinary mathematics community. In this way, observed patterns in Ellen's classroom were able to be better understood by studying her students' behavior and studying Ellen's perspectives on her own teaching.

What follows is a case study of Ellen's first *Mouse and Elephant* lesson on perimeter. The lesson is representative of her teaching throughout the three weeks of *Mouse and Elephant*, particularly her use of dialogic mathematical discourse.

Teaching Perimeter

After reviewing *Mouse and Elephant*, Ellen began teaching her students about perimeter, which is the exact number of (linear) units required to go around (or surround) a figure. *Mouse and Elephant* uses the representation of square unit tiles with the edge of each unit tile being of length 1. Therefore, the perimeters of the two rectangles shown below are 4 units and 6 units, respectively (see Figure 1):

Figure 1. 1x1 and 1x2 rectangles.

Following the *Mouse and Elephant* script for *Activity 1: Area and Perimeter*, Ellen distributed 12 tiles to each of her

students and then introduced the representation of perimeter as "the distance around a figure"[4] with the unit of measurement being the edge length of the unit tile. In particular, based on the "teacher talk" script she planned from in *Mouse and Elephant*, Ellen placed tiles on the overhead to make these rectangles (see Figure 2):

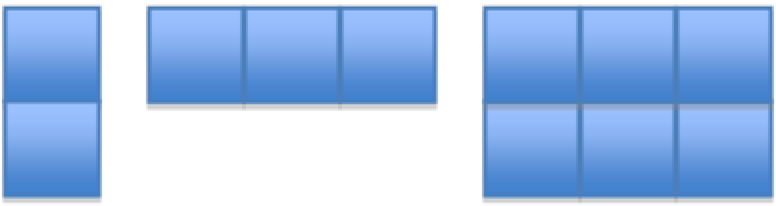

Figure 2. Ellen's Examples of Rectangles

Addressing the class, Ellen directed her students to use their tiles to construct the above rectangles at their desks and then find the perimeter of each:

Ellen: OK, so let's look at the first rectangle (*pause*). What is the perimeter?

Keisha[5]: (*raises hand, is called on by Ellen*) It's six!

Ellen: And how did you figure that out?

Keisha: I counted the edges of the tiles all the way around.

Ellen: Does everyone agree with Keisha (*nods from many students*)? Does anyone disagree (*students shake their heads no*)? OK, then how about the next rectangle, what is its perimeter (*pause*)?

Jack: (*raises hand, is called on by Ellen*) It's eight and it's eight because it's eight all the way around with the edges.

Ellen: Thank you Jack. Did anyone else get the same perimeter as Jack (*lots of hands go up*)? Marcie, could you come up and show us how you found the same perimeter as Jack (*she nods*)?

Marcie: (*comes up to the overhead and begins counting tile edges with a finger*) One, two, three, four, five, six, seven, and, uh, eight (*smiles as classmates nod in agreement*)!

Ellen: Jack, is that how you found the same perimeter

(*Jack nods*)? OK, thank you Marcie (*Marcie returns to her seat*)! How about the last rectangle, who would like to share their answer for the perimeter of the last rectangle (*lots of hands go up*)? Max, what did you get?

Max: Eight (*some students look puzzled*)!

Ellen: Eight (*Max nods*)? Max, could you come up and show us how you found the perimeter of eight (*Max nods and comes up to the overhead*)?

Max: (*standing at overhead and counting tile edges with a finger*) One, two, three, four, five, six, seven, eight (*pauses*) – uhhh, I forgot to count the last side (*looks up at Ellen*).

Ellen: That's OK, you can change your answer.

Max: Then it would be uhhh (*puts finger back on rectangle*), eight, then nine, and ten (*pauses, looking at rectangle and quickly counting tile edges again with finger*), so yea, it would be ten!

Ellen: What would be ten?

Max: The perimeter – the perimeter's ten.

Ellen: Does everyone agree with that (*pause as all students nod yes*)? OK – and thank you to Max for showing us how we can check our answer and then change it if it doesn't check (*short pause as Ellen leads quick applause for Max, who smiles and returns to his seat*).

Ellen's class continued for the next 12 minutes with Ellen distributing, and her students completing, a *Mouse and Elephant* practice sheet to build proficiency with making rectangles with tiles and finding the perimeter. In this practice exercise, students place tiles on rectangle outlines drawn on the sheet to physically make the rectangles with tiles, find the perimeter by counting the unit tile edges, and then record the perimeter for each. Students quickly completed the sheet as Ellen circulated around the classroom, and Ellen then called on different students to share their answers for the perimeter of each rectangle (all were correct with no disagreements from the class). Ellen then reminded her students to write their names on the sheet and place it in their math folders, which they did.

For the next segment of her lesson, Ellen placed tiles on the overhead to make these two figures (see Figure 3):

Figure 3. 1x1 and 2x2 squares.

As Ellen explained in a conversation after the lesson, her intent in displaying the 1x1 and 2x2 squares was to quickly summarize perimeter and then introduce the concept of area as the exact number of square units needed to cover a figure. Her selection of the two squares, she explained later, was to emphasize that area is measured in square units, with the area of the unit square tile being 1 square unit (i.e., the 1x1 square can be used as a unit of area to measure the area of the 2x2 square). With students' attention focused on her at the overhead, Ellen addressed her class:

Ellen: What are the perimeters of these two squares?

Kate: (*raises her hand and is called on by Ellen*) They're both four units!

Ellen: (*pauses*) And why do you think both have perimeter of four?

Kate: Because they're both squares and squares always have equal sides.

Ellen: (*pause*) What do others think? Are the perimeters of both squares four units? (*about half the students nod in agreement, the other half of the class looks unsure*). OK, would someone else like to explain your answer?

Josh: (*raises his hand and is called on by Ellen*) Kate's right and the perimeter is four because they're both squares – this has perimeter four (*holds up one tile*)

and if it were blown up bigger like the other one it would still have four sides so it would have four sides so the perimeter's four.

Chen: (*raises her hand and is called on by Ellen*) I think the small one is four and the big square is eight (*several other students nod in agreement*).

Ellen: Chen, why do you think that?

Chen: Because (*pause*) – you count up the edges around it and there are eight (*several other students nod in agreement*).

Ellen: Chen, could you come up and show us (*Chen comes up to the overhead and counts four tile edges around the 1x1 square and then counts 8 unit tile edges around the 2x2 square*)? OK – who agrees with Chen (*about two-thirds of the students raise their hands*)? Thank you Chen (*Chen returns to her seat*). (*pause*) So, we still need to decide if the perimeter of both rectangles if four or if the small one is four and the large one is 8.

Josh: (raises hand and is called on by Ellen) But they're not rectangles, they're squares!

Ellen: (*glances quickly at the clock*) Well (*pause*), squares are rectangles, they're just rectangles that happen to have all the sides the same length – do you all remember that (*some students nod, some look confused*)? Hmmm… (*glances again at clock, pauses*). OK, let's think of it this way (*Ellen walks to her desk, rummages in a drawer for a moment, then pulls out a small ball of string and a pair of scissors, and walks back to the overhead*). Mary, could you come up and help me (*Mary nods and walks up to the overhead from her desk cluster as Ellen unspools and cuts off a long piece of string*)? Hold the end of the string here (*Mary holds down the string at one corner of the 1x1 square as Ellen carefully wraps the string around the 1x1 square and then cuts off the segment; Mary and Ellen repeat this procedure for the 2x2 square and now have two pieces of string, one about twice as long as the other*). Now, we wrapped this shorter piece

around the small square, so isn't this the perimeter of the small square (*some nods*)? If we measure this with the edge of our tiles, it would match the perimeter, right (*more nods – Ellen then puts the piece of string on the overhead and shows that it's about as long as four edges by putting the string with four tiles lines up*) (Figure 4). Now, what about the big square (*Mary and Ellen repeat the procedure with the 2x2 square and line up tiles to show that the perimeter/string is about the length of eight tile edges*)? (Figure 5).

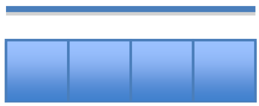

Figure 4. String and perimeter 1x1 square.

Figure 5. String and perimeter 2x2 square.

The conversation terminated with the following dialogue:

Ellen: So, the short string is the same length as the perimeter of the small square and the longer string is the same length as the perimeter of the big square, and the strings aren't the same, are they (*almost all students shake their heads no*)? Good, so the perimeter of the small square is (*pause*) four units (*points to the short string*) and the perimeter of the big square is (*points to the longer string*) eight units – are we all OK with that (*students nod and Ellen glances at the clock again*)? Now, Kate and Josh are right that the sides of a square are always the same, right (*all students nod*)? But if you have two different squares, each square has its sides the same, but one square can still have a larger perimeter (*students nod again*).

Kate: (*raises her hand and is called on by Ellen*) So, the squares have the same sides to themselves, but

(*pause*) the bigger square has bigger sides, so it has more perimeter.

Ellen: What does everyone think – do you agree with Kate (*all students nod*)? Josh, you agree too (*Josh nods*)? OK – Josh, could you show us how to count the perimeter of each square – without using the strings – like we did on the practice page?

Josh: (*nods, comes up to the overhead projector, and points at the unit tile edges as he counts*) So (*pause*), one, two three, four is the perimeter of this one (*points to the 1x1 square*) and one, two, three, four, five, six, seven, eight is the perimeter of the big one (*i.e., the 2x2 square*).

Ellen: Excellent Josh! (*Josh looks pleased and sits back in his seat at his desk cluster*) OK – we'll stop here on perimeter for today because it's almost time to go. Bring your tiles up and put them in the bin and we'll continue tomorrow.

Ellen's lesson from the first activity of *Mouse and Elephant* ended here, with the bell ringing for the end of class about two minutes later. After her students had filed out of her classroom, Ellen was able to reflect on her lesson and share with me how she felt it had gone.

Reflecting on the Perimeter Lesson

Ellen said that the perimeter lesson had gone "pretty well," though she was surprised at Kate's response that the perimeters of both the 1x1 and 2x2 squares were the same and both equal to four units. Elaborating, she said:

> We were just sailing right along and these kids were coming up with good ideas. (*pause*) But how can you plan for their misconceptions [like Kate's]? That's part of the problem with teaching a problem solving and reasoning and explaining way – you don't know what they're going to come up with. (*pause*) I think we ended in a good place and the kids get perimeter, but we only got through half of what I had hoped for – we didn't even get to area!

I asked Ellen about how she thought her students performed in terms of explaining their reasoning. She said that, “I was very pleased with how they explained themselves (pause) I would say that, yes, that was the best part of the lesson, that the kids were able to explain their thinking and reasoning well and I didn’t have to tell them answers.” I then asked Ellen about the way she used the string to help her students rethink the idea that the perimeters of the 1x1 and 2x2 squares were the same:

Author: So how did you decide to use the string – was that planned?
Ellen: Oh no (*laughs*)! I didn’t have that planned, but I did remember an activity that I did last year using string to measure the circumference of circular things, like cans and trash cans and lids, and I remembered too that I saw some string in my desk drawer this morning when was looking for something else. So I just thought, off the cuff, measure the perimeters of the squares with the strings, just like we measured the perimeters of the circles (*pause*) it was just another way of finding the perimeter and seeing that they were different.
Author: So how do you think the string worked – I mean comparing the perimeters of the 1x1 and 2x2 squares?
Ellen: Oh, I think it worked great – they could see that their original thinking wasn’t true and revised their idea and found the right perimeters for each, and they could explain why!

Finally, I pointed out to Ellen that in one part of her perimeter lesson, rather than help students unpack their reasoning to refine or revise their thinking, she corrected a misconception by telling her students the correct relationship. Specifically, when some students felt that squares were not rectangles, Ellen told them that squares are rectangles, that they just happen to be rectangles where all the sides are equal (as opposed to just opposite sides being equal):

Ellen: Yes, I just told them that squares are rectangles

because I didn't think we had time to go through that and resolve the issue that they're thinking that all squares have the same perimeter.

Author: So, it was mainly the time crunch you were thinking about in deciding to tell them that squares really are rectangles?

Ellen: Yes – if we had more time, we could do a (pause) like a two column chart and contrast the properties of squares and rectangles and then see that squares are just special rectangles, but I just didn't think that we had the time for that (pause) that could be a whole separate lesson!

Importantly, although Ellen chose not to explore the relationship between squares and rectangles in her perimeter lesson, she described an approach by which she could investigate it with her students in a way that is commensurate with how she and her students explored perimeter. Ellen telling her students that squares are rectangles, therefore, reflects a classroom reality of time constraints rather than an unwillingness or inability to engage students in reasoning and problem solving with rectangles and squares through dialogic mathematical discourse.

Key Findings about Mathematical Discourse in Ellen's Classroom

Four distinct themes emerged from analyzing Ellen's perimeter lesson: (a) she used *Mouse and Elephant* and divergent questions to support dialogic mathematical discourse; (b) despite the uncertainty of the outcome, using students' misconceptions about perimeter as a vehicle for developing greater understanding; (c) orchestrating teaching and learning in her classroom as a disciplinary community; (d) telling students a fact as a means of coping with time constraints. Each of these themes contributes to unpacking the dynamics of Ellen's classroom. Moreover, teasing out the four themes has the potential to support other teachers in implementing mathematical discourse in their own classrooms.

Dialogic Mathematical Discourse Supported by the Curriculum and Divergent Questions

A key finding of this case study is that Ellen implemented dialogic mathematical discourse in her classroom as is consistent with her stated goal of getting her students to talk about mathematics. Furthermore, this research shows that Ellen using a curriculum unit that supports mathematical discourse and her extensive use of divergent questions was integral to the mathematical discourse of the perimeter lesson, helping her students share and revise their ideas.

In analyzing Ellen's perimeter lesson, it was clear that her approach to classroom discourse is dialogic – i.e., she interacts with her students so that they are involved in a mathematical conversation as a way of engaging in mathematical reasoning and articulating their ideas about the mathematics (see Truxaw & DeFranco, 2008). Moreover, her dialogic approach is consistent with the classroom discourse delineated in the *Mouse and Elephant* unit on which she based her perimeter lesson (i.e., see earlier discussion of the teacher action, teacher talk, expected response structure of the teacher's role in *Mouse and Elephant*). Ellen's students played a critical role in the dialogic discourse of the lesson by contributing and refining their ideas, responding to the mathematical ideas of others, and engaging with Ellen. In this way, Ellen's perimeter lesson is a solid example of what dialogic discourse in the mathematics classroom can look like in terms of the role of the teacher, the role of the students, and also the role of the mathematics itself.

The mathematics content Ellen taught was, very importantly, not only about determining the perimeter of rectangles, but also about explaining how perimeter is determined and why a solution/answer does or does not make sense. Integral to the dialogic mathematical discourse in Ellen's classroom are mathematics and expectations for students' learning of the mathematics that support mathematical reasoning, which are, in turn, supported by the *Mouse and Elephant* unit on which Ellen based her lesson. Teacher, students, and the unit that inspired her lesson all factor into the dialogic mathematical discourse of Ellen's classroom. The perimeter lesson also shows that Ellen exhibits

dispositions and patterns of practice that support mathematical discourse in the classroom – e.g., Ellen typically asks divergent questions that encourage students to share their ideas and generate multiple responses, as opposed to convergent questions that often push students toward providing a single correct answer that is validated by the textbook or teacher. When Ellen does ask convergent questions, such as finding the perimeter of a given rectangle, explaining how the answer was found is expected, not just the answer. Moreover, Ellen's students know that their ideas and answers can be revised in light of new information or reasoning. Ellen is open to her students' ideas, and she supports them in being open to each others' ideas as well (see NCTM, 1991, 2000).

Although not all of her students participated in the dialogic mathematical discourse of the perimeter lesson, Ellen tries to make sure that all of her students contribute to class discussions periodically and monitors all of her students' understanding through homework. During a different lesson, for example, Ellen noted that it is often not possible for every student to participate in the discussion for every lesson. However, she says that she "keeps track" of students who are reluctant to participate and works to help them become comfortable sharing their ideas in class. Ellen reports that some students prefer to show their work and then answer questions about it rather than describing their ideas in words only. Ellen shared that, "Getting my kids comfortable talking and sharing – and it's OK to do that in different ways, maybe not every day, but two or three times a week – is really important." This is another feature of Ellen's practice that supports dialogic mathematical discourse in her classroom.

A Misconception: Uncertainty for the Teacher, Opportunities for Students

A second finding of this case study is that Ellen's adherence to maintaining dialogic mathematical discourse in her classroom included students entering unexpected mathematical territory, requiring Ellen to go with them. Facilitating mathematical discourse with her students to work through and revise their ideas, although uncertain for Ellen,

resulted in opportunities for her students to talk about and wrestle with the mathematics (in this case, perimeter). In her reflection on the lesson, Ellen felt that although it was challenging to her as a teacher, the learning opportunities the lesson offered her students as part of the mathematical discourse were well worth it.

Ellen’s students proposed, and many initially supported, a misconception about the perimeter of squares. In her teaching, even though she was surprised by her students’ misconception, Ellen did not tell her students they were wrong nor did she demonstrate a valid procedure (e.g., by counting the tile edges around each square) to calculate the correct perimeters. Instead, she continued teaching through dialogic mathematical discourse and reframed the problem (i.e., using the string as a tool to measure the perimeter) to allow her students to explore it further. By working with the misconception, rather than attempting to ignore or work around it, Ellen engaged her students in mathematical reasoning (NCTM, 2000) and what Bruner (1960) characterized as “intellectual honesty,” whereby the work of students in the classroom reflects the actual discipline of mathematics where conjectures are developed, then studied and scrutinized, and then may be proven false, resulting in deeper understanding (see Lakatos, 1995). As described by researchers investigating mathematical discourse as a component of teaching mathematics for understanding, Ellen’s perimeter lesson reflects both uncertainties for the teacher and rich mathematical opportunity for the students (e.g., Ball, 1990; Clark, 1997; Kisker et al., 2012; Rickard, 2005a, 2005b; Stein et al., 2000).

The Mathematics Classroom as a Disciplinary Community

A third finding emerging from the study is that the way in which Ellen engaged her students about the perimeter of squares misconception (i.e., that the perimeter of any square is four because all of the sides are equal) sheds further light on how her classroom instruction parallels actual disciplinary work in mathematics. Lakatos (1995) showed how mathematical ideas may be analyzed through an interactive process where the mathematics is illustrated with one or more

examples, subjected to testing through the development of counterexamples that disprove or "falsify" the conjecture, and then the idea is revised and tested again, ultimately leading to mathematics that may be proven, or to the conclusion that the original idea was invalid. In Ellen's classroom, when she identified her students' misconception about the perimeter of squares always being four, she asked for student volunteers to clarify their reasoning. This resulted in uncovering some students' thinking that because squares have four equal sides, they always have a perimeter of four units. As these students explained their reasoning, Ellen also encouraged students who had different ideas, such as the notion that squares' perimeter should be measured in the same way as rectangles, to explain their thinking as well. From there, Ellen introduced the string as a means of testing both ideas, which resulted in her class (correctly) determining that all squares do not have perimeter of four units, refuting the misconception and affirming that the perimeter of squares can be determined in the same way as the perimeter of any rectangle.

This analysis demonstrates that Ellen's experience as an effective mathematics teacher reveals itself not in avoiding misconceptions or always resulting in orderly mathematics, but in engaging her students in processes of doing mathematics and wrestling with mathematical concepts to construct understanding. Ellen is able to engage her students in dialogic mathematical discourse so that her students have important mathematical experiences whether grappling with a misconception or not.

Telling Students a Fact: Time Constraints in the Classroom

An additional finding of this case study is that, even with her years of experience and a demonstrated ability to connect her students with mathematics, Ellen is not immune to time constraints in her classroom. In particular, near the end of the perimeter lesson, after the misconception about the perimeter of squares had been uncovered and time was running short, Ellen told her students that all squares were rectangles and continued on assuming this fact, even though some of her students were not completely sure of it. As noted earlier, Ellen

acknowledged this in her reflection and cited time running out as why she simply told (or reminded) her students that squares are really rectangles that just happen to have all sides equal. Although she could have pursued an investigation of the properties of rectangles and squares to show that squares are rectangles (e.g., compare and contrast properties to see that squares have all the properties of rectangles and, therefore, are rectangles), Ellen chose not to, making the instructional decision that it was more important to address the misconception about the perimeter of squares. Making such instructional decisions are part of teaching, including teaching mathematics in ways that engage students in dialogic mathematical discourse. What is also notable about this decision on Ellen's part is that, not only is she aware that she "told" students about this mathematical relationship between rectangles and squares, but that she knows how she could engage her students in exploring the relationship if time permitted. This further suggests that engaging her students in dialogic mathematical discourse to learn mathematics is systemic in her classroom.

Conclusions

This case study of Ellen Wilson's teaching about perimeter is typical of her teaching with the entire *Mouse and Elephant* unit and shows that the middle school classroom can be fertile ground for dialogic mathematical discourse. Ellen's lesson demonstrates what dialogic mathematical discourse in middle school mathematics can look like, how such teaching can take a teacher and her students in unexpected directions, and how the teacher can engage her students as participants in doing mathematics by developing, articulating, and refining their mathematical ideas. Consistent with prior research, the dialogic discourse in Ellen's classroom is characterized not only by engaging her students, but also by combining the mathematics and supporting curriculum materials to craft an overall experience where students are participants in mathematical activity that parallels authentic work in the discipline, such as formulating, testing, and revising conjectures and employing counterexamples (cf., Ball, 1990;

Lakatos, 1995; NCTM, 2000; Rickard, 1996, 1998; 2005a, 2005b). Ellen's classroom shows that, consistent with the recommendations of reforms, experiencing and learning about reasoning and the process of doing mathematics along with specific content (e.g., determining the perimeter of rectangles) is valuable (e.g., NCTM, 1991, 2000). Moreover, Ellen and her students provide evidence that misconceptions may serve as opportunities for thoughtful mathematical discourse and can yield important mathematical experiences for students.

The above conclusions that may be drawn from this case study also imply multiple challenges that Ellen and other teachers may encounter employing dialogic mathematical discourse in the middle school classroom. For example, even an experienced teacher like Ellen bumps up against the realities of time constraints and the limitations of planning to anticipate all aspects of how students will engage with the mathematics. As well as telling her students that "squares are rectangles" rather than exploring the relationship due to time running out, Ellen also dealt with the unexpected misconception that that all squares have a perimeter of four because all four sides are equal. In this case study, Ellen addressed these challenges in different ways – i.e., she knows that she could return to the "squares are rectangles" assertion and actually investigate this with her students using the same kind of dialogic mathematical discourse which she used to successfully address the perimeter of squares misconception.

But not all middle school mathematics teachers necessarily have the knowledge and dispositions to deal flexibly with time constraints, address planning that goes awry due to unexpected student misconceptions, and orchestrate dialogic mathematical discourse as successfully as Ellen does in her classroom. Therefore, helping teachers develop knowledge and dispositions to effectively implement mathematical discourse in their classrooms remains an ongoing challenge for preservice teacher education and professional development for inservice teachers (Bell, Wilson, Higgins, & McCoach, 2010; Clark, 1997; Conklin et al., 2006; Lamberg, 2013; Rickard, 1996; Stein et al., 2000; Vacc & Bright, 1999). Studying the practice of teachers like Ellen holds potential for teachers'

professional development to implement dialogic mathematical discourse in their own classrooms.

Another implication of this case study is the need to further investigate how and to what extent a teacher's ability to explore and reason about mathematics with her students through dialogic mathematical discourse impacts her students' mathematics achievement. For example, did the experience of reasoning through how to find the perimeter of rectangles and the misconception about perimeter of squares support (or hinder?) increased mathematics achievement for Ellen's students? How might such change in mathematics achievement be measured – e.g., by accuracy in computing perimeter, ability to explain solutions and reasoning, ability to solve/explain nonstandard problems, all of these? Although research has provided evidence, for example, that teachers' content knowledge and dispositions may impact students' mathematics achievement (e.g., Hill et al., 2005; Rickard, 1995), research on the impact of dialogic classroom discourse is less extensive (e.g., Herbal-Eisenmann & Otten, 2011; Lamberg, 2013).

Ellen and her students yield insights into how dialogic mathematical discourse can be implemented in the middle school classroom. The work of Ellen and her students shows how teacher and students can explore mathematical terrain together, bumping up against unexpected ideas and new mathematics, respectively. All contribute to articulating, sharing, reexamining, and revising their ideas about mathematics, which is at the heart of a community of mathematics learners. However, none of this happens automatically or easily. Creating and maintaining a disciplinary community using mathematical discourse places pedagogical demands on the teacher. It requires a high level of knowledge of mathematics and dispositions for orchestrating mathematical discourse in the classroom, including fostering a classroom climate where students are comfortable enough to articulate, test, and perhaps revise their ideas in a public setting. Supporting middle school teachers to be able to implement dialogic mathematical discourse in their classrooms, like Ellen did, will continue to require ongoing effort from both preservice teacher education and inservice professional

development as recommended by reforms (e.g., NCTM, 1991, 2000) and further investigation into how dialogic mathematical discourse can impact students' mathematics achievement.

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1 Ellen's middle school is in an urban neighborhood in a medium-sized Midwestern city; her school enrolled about 1,000 students in grades 6-8 at the time of this study with about 50% of the student body Caucasian, about 30% African American, and about 20% Hispanic and Asian.

2 The activities were from *Factors and Multiples* (Fitzgerald, Winter, Lappan, & Phillips, 1986), which focuses on prime and composite numbers, factors, multiples, and divisors.

3 The title *Mouse and Elephant* refers to the central challenge of the unit, which can be summarized as: Suppose a mouse is represented by a unit cube, and an elephant is represented by a cube that is as high as 40 mice – if the area of the mouse's coat is equal to the surface area of the unit cube, how many mouse coats are needed to make an elephant coat? Solving this challenge requires students to learn about perimeter, area, surface area, and volume, and the relationships between these measures.

4 Ellen based her lesson on perimeter on the first activity in the *Mouse and Elephant* unit, but decided to not follow the script exactly; she wanted to pace her lesson more slowly than the unit, but her use of the tiles is consistent with the unit.

5 All students' names are pseudonyms. In choosing pseudonyms I've attempted to represent the diversity of Ellen's class.